Stochastic Manufacturing & Service Systems

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Newsvendor Model (cont’d)

1 Cost Minimization Problem

**Solution to Cost Minimization Problem** By using a similar method to solve the profit maximization problem, we have the following useful theorem.

**Theorem 1.** Let $D$ be the random demand with c.d.f. $F(x) = \mathbb{P}(D \leq x)$ for all $x \in (-\infty, \infty)$. If $D$ is a continuous rv, then the optimal production quantity that solves the cost minimization problem is $y^*$ such that

$$F(y^*) = \frac{c_u - c_v}{c_u + c_o}.$$

If $D$ is a discrete rv, then the optimal production quantity that solves the cost minimization problem is the smallest $y^*$ such that

$$F(y^*) \geq \frac{c_u - c_v}{c_u + c_o}.$$ 

I would like to give several interesting remarks:

1. Fixed cost ($c_f$) again does not affect the optimal production quantity.
2. If understock cost ($c_u$) is equal to unit production cost ($c_v$), which makes $c_u - c_v = 0$, then you will not produce anything.
3. If unit production cost and overstock cost are negligible compared to understock cost, meaning $c_u \gg c_v, c_o$, you will prepare as much as you can.

2 Cost Minimization with Initial Inventory

**Example 2.1.** Let $c_u = 10, c_v = 4, c_o = 2, c_f = 10$. The initial inventory level $x = 13$ units. $D$ has a discrete uniform distribution on $\{11, 12, \ldots, 19, 20\}$. Here I have two questions for you:

1. Should you produce?
2. If so, how much to produce to minimize expected total cost?
Solution. Let $y$ be the produce-up-to-quantity. By “produce-up-to-quantity”, we mean that we produce $y - x$ so that our inventory at the beginning of the next period becomes $y$. Suppose $y > x = 13$.

\[
E[cost(y, D)] = c_v(y - x) + c_f + c_o E[(y - D)^+] + c_u E[(D - y)^+]
\]

\[
= -c_v x + c_f + c_v y + c_o E[(y - D)^+] + c_u E[(D - y)^+]
\]

Since $-c_v x, c_f$ does not depend on $y$, $y^*$ is the same as before. Therefore, “if” we decide to produce up to $y^*$,

\[
F(y^*) = \frac{10 - 4}{10 + 2} = \frac{6}{12} = \frac{1}{2}
\]

\[
y^* = 15
\]

Question: Can we conclude that for $x = 13$, we produce 2 items to bring initial inventory to produce-up-to-quantity, 15?

If $c_f$ is one million dollars, you probably do not want to produce anything. You will just want to pay the penalty understock cost instead of paying the fixed production fee. Then, how do I know if I need to produce or not. In this case, you need to evaluate the total cost. Here are two scenarios: to produce (then I should produce 2 items) or not to produce.

(1) If you do not produce, the expected cost is

\[
E[cost(x, D)] = c_u E(D - x)^+ + c_o E(x - D)^+
\]

\[
E(D - 13)^+ = 1 \frac{1}{10} + 2 \frac{1}{10} + \cdots + 7 \frac{1}{10} = \frac{28}{10} = \frac{14}{5}
\]

\[
E(13 - D)^+ = 2 \frac{1}{10} + 1 \frac{1}{10} = \frac{3}{10}
\]

\[
E[cost(x, D)] = 10 \frac{14}{5} + 2 \frac{3}{10} = 28 + 0.6 = 28.6.
\]  

(2) If you do produce 2 items more, i.e. $y^* = 15$, the expected cost is

\[
E[cost(x, y^*, D)] = -c_v x + c_f + c_v y^* + c_o E(y^* - D)^+ + c_u E(D - y^*)^+
\]

\[
= 4(15 - 13) + 10 + 2E(15 - D)^+ + 10E(D - 15)^+
\]

\[
E(15 - D)^+ = \frac{4 + 3 + 2 + 1}{10} = 1
\]

\[
E(D - 15)^+ = \frac{1 + 2 + 3 + 4 + 5}{10} = 1.5
\]

\[
E[cost(x, y^*, D)] = 18 + 2(1) + 10(1.5) = 35.
\]

Since (1) is less than (2), you should not produce.

3 Continuous Demand

Now, move on to continuous random demand case. Remember that $c_u, c_v, c_o, c_f$ denote understock cost, variable cost, overstock cost and fixed cost respectively.
Example 3.1.  •  \( D \sim \text{Uniform}[10, 20] \)

•  \( c_u = \$10, \ c_v = \$4, \ c_o = \$2, \ c_f = \$30. \)

You are asked to answer the following three questions:

(a) What is the optimal produce-up-to quantity to use in this case?

(b) Suppose that \( x \) is the initial inventory level. Do you produce?

(c) What is the critical value \( x^* \) such that if \( x \leq x^* \) you produce otherwise you do not.

1. optimal produce-up-to quantity \( y^* \) satisfies

\[
F(y^*) \geq \frac{c_u - c_v}{c_u + c_o} = \frac{10 - 4}{10 + 2} = \frac{1}{2} = \frac{y^* - 10}{20 - 10},
\]

which implies \( y^* = 15. \)

2. Suppose that \( x \) is the initial inventory level. For initial stock, it is assumed that you do not have to pay for storage costs, etc. If \( x \geq 15 \), you already have more than optimal level of inventory, so you do not produce any items.

Assuming \( x = 10 \), evaluate different scenarios of actions.

(a) What if you decide not to produce anything? Then, it means that you will be using the existing items to fulfill the orders from your customers. You are currently holding 10 items. It was inherited to you, so you do not have to pay for purchasing. Your cost structure in this case is composed of understock cost and overstock cost.

\[
\mathbb{E}[\text{Cost}(x, D)] = \mathbb{E}[\text{understock cost}] + \mathbb{E}[\text{overstock cost}]
= (10)\mathbb{E}(D - 10)^+ + 0,
\]

where

\[
\mathbb{E}(D - 10)^+ = \int_{-\infty}^{\infty} (x - 10)^+ f(x)dx
= \int_{10}^{20} (x - 10)^+ f(x)dx
\]

\[
\therefore f(x) \text{ has non-zero value only within } [10, 20] \text{ range.}
\]

\[
= \int_{10}^{20} (x - 10) f(x)dx
\]

\[
\therefore \text{ We can safely remove the plus sign because the integration range is } [10, 20].
\]

\[
= \frac{1}{10} \left( \frac{1}{2} \right) (x - 10)^2 \bigg|_{10}^{20} = 5
\]

(b) What if you decide to produce? You know that you should produce 5 gallons to set your inventory level to the optimal point, 15 gallons. In this case, your cost components are purchasing cost, fixed ordering cost, understock cost, and holding cost. Here, you should
have felt that total cost would exceed the previous case because fixed ordering cost $30 seems to be too much. Anyway, let us proceed.

\[ E[\text{total cost}] = E[\text{purchasing cost}] + E[\text{fixed cost}] + E[\text{understock cost}] + E[\text{overstock cost}] \]

\[ = 4(5) + 30 + 10E(D - 15)^+ + 2E(15 - D)^+ \]

\[ = 50 + \frac{5}{4} + \frac{5}{4} = 50 + 15 = 65 \]

Make sure that you can compute the following.

\[ E(D - 15)^+ = \int_{-\infty}^{\infty} (x - 15)^+ f(x)dx \]

\[ = \int_{15}^{20} (x - 15) \frac{1}{10} dx = \frac{1}{10} \frac{1}{2} (x - 15)^2 \bigg|_{15}^{20} = \frac{1}{20} \times 5^2 = \frac{5}{4} \]

We have just established ordering policy if inventory level is 10. You should give consistent and exhaustive instructions to your assistant on producing items at the end of Friday. We can also set up the scenario for \( x = 0 \). The optimal action (suspected) for that case would be to produce 15 gallons. This is not sufficient as a policy. At this stage, some of you have figured out that there is a critical value \( x^* \) such that if \( x \leq x^* \) you produce otherwise you do not.

How do we compute \( x^* \)? One thing you can immediately think of would be obtaining the point \( x^* \) where producing cost equals to not producing cost. Before jumping in, we know that the critical value should be somewhere between 0 and 10 i.e. \( 0 \leq x^* \leq 10 \). Why? Because when we previously concluded that if \( x = 10 \) we should not produce and if \( x = 0 \) we do need to produce.

Let \( 0 < x < 10 \).

(a) This is the case you produce. What would be the expected total cost?

\[ E[\text{total cost}] = 4(15 - x) + 30 + 10E(D - 15)^+ + 2E(15 - D)^+ \]

\[ = 4(15 - x - 5) + [4(5) + 30 + 10E(D - 15)^+ + 2E(15 - D)^+] \]

\[ = 4(10 - x) + 65 \]

(b) When you do not have to produce, what would be the expected total cost?

\[ E[\text{total cost}] = 10E(D - x)^+ + 0 = 10E(D - x) = 10[E(D) - x] = 10(15 - x) \]

For \( x^* \), these two costs should be equal.

\[ 4(10 - x^*) + 65 = 10(15 - x^*) \]

\[ 40 - 4x^* + 65 = 150 - 10x^* \]

\[ 6x^* = 150 - 105 = 45 \]

\[ x^* = \frac{45}{6} = \frac{15}{2} = 7.5 \text{ gallons} \]